ON A CERTAIN CLASS OF TRANSFORMS

BY

BENJAMIN EPSTEIN

S.B., Massachusetts Institute of Technology, 1937
M.S., Massachusetts Institute of Technology, 1938

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1941

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Introduction: In 1937 Bourgin and Duffin wrote a paper entitled "The Heaviside Operational Calculus" in which they discussed the integral equation

$$(1.)\quad \psi(s) = \int_A^\infty \int_0^\infty v^{s-1} e^{-v\lambda}\, \varphi(v)\, f(\lambda)\, dv\, d\lambda, \quad A \geq 0.$$

Equation (1.) may be considered as the two integral equations

$$(2.)\quad \psi(s) = \int_A^\infty K(s,\lambda)\, f(\lambda)\, d\lambda$$

and

$$(3.)\quad K(s,\lambda) = \int_0^\infty v^{s-1} e^{-v\lambda}\, \varphi(v)\, dv.$$

In case $\varphi(v) = 1$, equation (1.) reduces to

$$(4.)\quad h(s) = \psi(s)/\Gamma(s) = \int_A^\infty f(\lambda)/\lambda^s\, d\lambda$$

i.e., the Mellin transformation. For the particular case $\varphi(v) = 1/1-e^{-v}$ they give a rigorous inversion formula if $f(\lambda)$ belongs to $L(A,\infty)$ and satisfies certain other conditions. Further they write down a formal solution for any given $\varphi(v)$. In this thesis we are interested in relating asymptotic properties of $\psi(s)$ as $s \to 1^+$ with those of $f(\lambda)$ as $\lambda \to \infty$. We are also concerned with moment and closure problems associated with equation (1.). Throughout most of our discussion $\varphi(v)$ will be equal to $1/1-e^{-v}$.

In Chapters I and II the asymptotic properties of $\psi(s)$ in equation (1.) are discussed. It is well-known that for the Laplace integral equation

$$(5.)\quad h(s) = \int_0^\infty e^{-s\lambda}\, f(\lambda)\, d\lambda$$

that it is important to connect the asymptotic behavior of $f(\lambda)$ at $0, \infty$ with the behavior of $h(s)$ as $s \to \infty$, 0 respectively. We considered for (1.) problems analogous to those discussed in the literature for (5.) and found that the iterated use of known theorems for Laplace or Mellin transforms led to results in this case. Also it is possible to show that the problem is not only an asymptotic one but can be generalized to one of Tauberian type.

In Chapter III we discuss a moment problem for equation (1.). The question may be stated as follows--- suppose $f(\lambda)$ is put into the transforming machinery of (1.) where $\psi(s)$ is given at an enumerable set of points, $\{s_n\}$, monotone increasing with limit infinity. Let the values of $\psi(s)$ at these points be denoted by $\{\mu_n\}$, $n = 1,2,\ldots$. Suppose a solution $f(\lambda)$ of the infinite number of equations does exist, then does there exist another function which has the same moments $\{\mu_n\}$? The problem then is to impose conditions on the $\{\mu_n\}$ which will ensure that there exists only one such solution.

For solving these problems, a theorem in a paper by R. P. Boas[3] and the Carleman theorem for meromorphic functions in a half-plane[4] were utilized. Further the problem was discussed in those cases where the solution was required not only to belong to a given function class, but further

satisfied certain growth conditions at the origin or infinity.

Many problems are still open in this thesis. One line of generalization consists in considering

$$(6.)\quad \psi(s) = \int_0^\infty \int_0^\infty v^{s-1} K(v\lambda)\ \varphi(v) f(\lambda)\, dv\, d\lambda$$

with $K(v\lambda)$ in place of $e^{-v\lambda}$. This is also an interesting class of equations and if we let $K(v)$ satisfy certain conditions such as going down exponentially at infinity, considerable information can be obtained relating the properties of $\psi(s)$ and $f(\lambda)$. Incidentally the method of Fourier transforms used in Chapters I and II are just as easily and readily applied in this case.

To give an indication of the results proved in this thesis, we select certain representative theorems from each chapter. In Chapter I, one of the theorems may be stated as follows:

Theorem I: Assume in (1.) that

$$(7.)\quad \psi(s) \sim C/(s-1)^\gamma \text{ as } s \to 1^+,\ \gamma \geq 0,\ C \text{ an arbitrary complex}$$

number. $\psi(s)$ is assumed to be convergent for $s > 1$.

$$(8.)\quad e^{-v\lambda} f(\lambda) \geq 0 \text{ and belongs to } L(0,\infty), \text{ all } v > 0.$$

Further let $f(\lambda) \equiv 0$ over an interval $0 \leq \lambda \leq \Delta$, $\Delta > 0$ and be monotone increasing over the interval $\Delta \leq \lambda \leq 1$, then

$$(9.)\quad \lim_{\tau \to \infty} \int_0^\tau f(\lambda)\, d\lambda \sim C/\Gamma(r)\ (\log \tau)^{r-1}$$

In Chapter II, one of the theorems may be stated as follows:

Theorem IV: If $f(\lambda)$ belongs to $L_2(A,\infty)$ in (2.) for $A > 0$ then $\psi(s)/\Gamma(s)$ belongs to L_2 along every line $\mathrm{Re}\, s = d > \frac{1}{2}$.

In Chapter III, one of the theorems may be stated as follows:

Theorem VII: Given two functions, $f_i(\lambda) \geq 0$ and integrable $L(0,\infty)$, further yielding the same $\psi(n)$ in (1.). Then a sufficient condition that $f_1(\lambda) \equiv f_2(\lambda)$ a.e. is that

$$(10.)\quad \log \psi(n) \leq 3\, n \log n + 2\, n \log \sigma(n), \text{ where } \lim_{r \to \infty}$$

$\log \sigma(r) = -\infty$ and $\log r + \log \sigma(r)$ is non-decreasing.

NOTES

1. D. G. Bourgin and R. J. Duffin: The Heaviside Operational Calculus, Am. Journal of Math., Vol. 59, No. 3, pp. 489-505. For the exact hypotheses of their inversion theorem see pp. 502-504.
2. For a discussion of Abelian and Tauberian theorems for the Laplace integral equation the reader may refer to G. Doetsch's book, Theorie und Anwendung der Laplace Transformation, 1937, Chapter 10. Here a number of results in the literature due to Karamata, Hardy, Littlewood, Wiener, and Doetsch are collected in systematic form.
3. R. P. Boas: On a Generalization of the Stieltjes Moment Problem, Trans. of the Am. Math. Society, V. 46, pp. 142-150.
4. E. C. Titchmarsh: Theory of Functions, 1932, pp. 130-132.

VITA

The author, Benjamin Epstein, was born on March 5, 1918, in Boston, Massachusetts. He attended the public schools in the City of Boston, and his secondary school work was taken in the Boston Public Latin School. He entered the Massachusetts Institute of Technology in September, 1934, and received the S.B. degree in June, 1937. He received the M.S. degree from the same institution in June, 1938. He came to the University of Illinois in September, 1938, as University Fellow for the year 1938-1939. From September, 1939, to June, 1941, he has been a teaching assistant in the Department of Mathematics. He is a member of Sigma Xi and the American Mathematical Society. He has published an article entitled: "Proof of a Theorem of Carleman," in the January, 1939, issue of the *M.I.T. Journal of Mathematics and Physics.*